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SWITCHING THEORY OF LINEAR MICROWAVE NETWORKS

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3.5 Linear Microwave Sources

3.51 Linear 1-Port Sources

Generators used in microwave engineering often deliver their energy via dominant-mode waveguides. If the single-frequency, or at least spectrally limited, signal generated here lies within the single-wave region of the dominant-mode line, a one-port source can be associated with the generator junction (cf. Section 3.3). In many cases, such a 1-port source can be approximated with the model of a linear 1-port source. By this we mean a 1-port network which is adequately described by one time-independent (sourceless) parameter and one source variable which is independent of it; the sourceless parameter and the source variable are themselves independent of the external circuitry, i.e., independent of the variables of state in the reference plane defining the 1-port network on the dominant-mode line.

In the following, we will limit our discussion to linear 1-port sources with sinusoidal (time-wise) source variables, i.e., single-frequency source variables whose frequencies lie within the domain of definition of the 1-port network. If this source is now connected with a linear, time-independent and sourceless 1-port network or an equivalent network, all variables of state for the entire system are sinusoidal and of the same frequency as the source variable, and can thus be represented by vectorial or scalar components. The operating state of the source is now described by a suitable pair of state variables in the reference plane, in the same manner as for the sourceless 1-port network. The same conventions are assumed to apply to the coordinate system and the reference structure as for the corresponding 1-port network; in particular, the z-direction is taken to be positive going into the junction in both cases. A suitable pair of variables of state for describing the operating state of the linear 1-port source, for example, are the vectors $\underline{E}^{+}_{t1}$, $\underline{E}^{-}_{t1}$ in the reference plane $z = 0$. In contrast to the linear time-independent and sourceless 1-port network, however, the departing wave $\underline{E}^{-}_{t1}$ is now no longer proportional to the arriving wave $\underline{E}^{+}_{t1}$ (cf. Eq. (3.4/1a)), but is a general linear function of two variables: the arriving wave $\underline{E}^{+}_{t1}$ and a source wave $\underline{E}^{-}_{tQ1}$, independent of it, with the same structure as $\underline{E}^{+}_{t1}$ and $\underline{E}^{-}_{t1}$. We can then describe the operating state, for example, with the equation of state

$$\mathbf{E}_{t1}^{-} = r_1 \mathbf{E}_{t1}^{+} + \mathbf{E}_{tQ1}^{-} \qquad (3.5/1)$$

We break down the departing source wave $\underline{E}^{-}_{tQ1}$ at point $z = 0$, corresponding to the vector quantities $\mathbf{E}_{t1}^{+} = \mathbf{t}_{1+}\sqrt{Z_{F1}}\, a_1$ and $\mathbf{E}_{t1}^{-} = \mathbf{t}_{1+}\sqrt{Z_{F1}}\, b_1$, into its vectorial and scalar components

$$\mathbf{E}_{tQ1}^{-} = \mathbf{t}_{1+}\sqrt{Z_{F1}}\, b_{Q1} \qquad (3.5/2)$$

and thus obtain the scalar equation of state from Eq. (3.5/1):

$$b_1 = r_1 a_1 + b_{Q1}. \qquad (3.5/3)$$

We call the quantity r_1 the internal reflection factor, or reflection factor for short, and b_{Q1} the departing source wave variable, or source wave or original wave for short, of the linear 1-port source.

If the source is closed off with a suitable ideal absorber, then, in accordance with Definition 11, departing wave $\underline{E}^{+}_{t1}$ or a_1 is zero, and the departing wave is thus equivalent to the source wave $\underline{E}^{-}_{t1} = \underline{E}^{-}_{tQ1}$, $b_1 = b_{Q1}$. Thus at the same time, we have a measurement rule for determining the departing source wave. In this configuration, the effective power delivered to the absorber by the source is

$$P_{p1}^{-} = \tfrac{1}{2}\,|b_1|^2 = \tfrac{1}{2}\,|b_{Q1}|^2. \qquad (3.5/4)$$

This is also the maximum deliverable or available effectivo power from a 1-port source with $r_1 = 0$ (see also Section 4.25).

Like sourceless 1-port networks, 1-port sources can also be represented in a different manner (cf. Fig.3/11). We obtain the voltage/current representation from the special wave representation (Eq. (3.5/3)) if we express the port wave variables a_1, b_1 with port voltage u_1 and port current i_1, using Eq. (2.5/13):

$$\tfrac{1}{2}(u_1 - i_1) = r_1 \tfrac{1}{2}(u_1 + i_1) + b_{Q1}.$$

By solving for $u_1(i_1)$ and $i_1(u_1)$ respectively, we then obtain

$$u_1 = z_1 i_1 + u_{Q1} \qquad (3.5/5a)$$

with

$$z_1 = \frac{1 + r_1}{1 - r_1} \quad (3.5/5b) \qquad \text{and} \qquad u_{Q1} = \frac{2 b_{Q1}}{1 - r_1} \quad (3.5/5c)$$

or

$$i_1 = y_1 u_1 + i_{Q1} \qquad (3.5/6a)$$

with

$$y_1 = \frac{1 - r_1}{1 + r_1} \quad (3.5/6b) \qquad \text{and} \qquad i_{Q1} = \frac{-2 b_{Q1}}{1 + r_1}. \quad (3.5/6c)$$

Representations (3.5/5a) and (3.5/6a) have long been known, in the engineering of concentrated circuits, as equations of state for the equivalent voltage source and for the equivalent current source, respectively[10].

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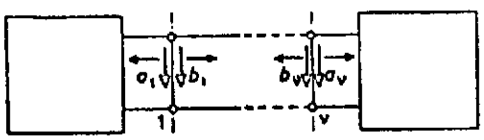
[10]The Equations for the equivalent voltage source and the equivalent current source are often given in the literature in the so-called "generator notation," i.e. with the current-arrow direction opposite to that in Fig. 3/11, while here "load notation" is used uniformly for all n-port networks. It should also be pointed out that reduced state variables (u_1, i_1, u_{Q1}, i_{Q1}) and normalized operator variables (z_1, y_1) are used here.

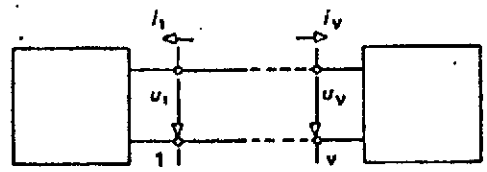
(a)



1-port source

Sourceless
1-port network

(b)



1-port source

Sourceless
1-port network

Equations of state for the	linear 1-port source
Scattering form	$b_1 = r_1 a_1 + b_{Q1}$
Inverse scattering form	$a_1 = v_1 b_1 + a_{Q1}$
Impedance form	$u_1 = z_1 i_1 + u_{Q1}$
Conductance form	$i_1 = y_1 u_1 + i_{Q1}$

Fig. 3/11. 1-port source and sourceless 1-port network represented in two-terminal form with load "metering" arrows. (a) Wave representation, (b) voltage / current representation.

The time-independent parameters z_1 and y_1 are internal impedance and internal conductance, respectively, and the source variables u_{Q1} and i_{Q1} are the source voltage or original voltage and the source current or original current, respectively. Making use of these concepts, Butterweck [14] proposed the name "equivalent wave source" for the source represented by the wave representation (3.5/3). It should be emphasized, however, that all three cases involve the same linear 1-port source, which is merely described by three different equations of state.

Characteristic parameters r_1 and b_{Q1} for the equivalent wave source are obtained from the characteristic parameters z_1 and u_{Q1} for the equivalent voltage source or y_1 and i_{Q1} for the equivalent current source by transforming Eqs. (3.5/5a) and (3.5/6a) into the "scattering form" Eq. (3.5/3), with the aid of

Eq. (2.5/12). We obtain

$$r_1 = \frac{z_1 - 1}{z_1 + 1} \quad (3.5/7a) \qquad \text{and} \qquad b_{Q1} = \frac{u_{Q1}}{z_1 + 1} \quad (3.5/7b)$$

$$r_1 = -\frac{y_1 - 1}{y_1 + 1} \quad (3.5/8a) \qquad \text{and} \qquad b_{Q1} = -\frac{i_{Q1}}{y_1 + 1}. \quad (3.5/8b)$$

In addition, the following relations exist between the characteristic parameters for the voltage/current representation:

$$z_1 y_1 = 1 \quad (3.5/9a) \qquad \text{and} \qquad \frac{u_{Q1}}{i_{Q1}} = -z_1 = -y_1^{-1}. \quad (3.5/9b)$$

There are three important special cases of the linear 1-port source: If $z_1 = 0$, then port voltage u_1 in Eq.(3.5/5a) is always equal to source voltage u_{Q1}. In this case, the source is also known as a primary voltage source. If $y_1 = 0$, then port current i_1 in Eq. (3.5/6a) is always equal to source current i_{Q1}. This case is called the primary current source. Making use of these concepts, we then call the equivalent wave source the primary wave source in the wave representation for the case $r_1 = 0$; the departing port wave b_1 is always equal to the source wave b_{Q1} here.

While the primary voltage source ($z_1 = 0$) and the primary current source ($y_1 = 0$) represent limiting cases which are difficult to effect in microwave engineering, the primary lead source ($r_1 = 0$) is easily realizable and is usually striven for in the designing of practical microwave generators. When we compare the three modes of representation, we recognize the formal superiority of the wave representation in the scattering form, Eq. (3.5/3) : While the limiting case $z_1 = 0$ cannot be represented in the "equivalent current source" conductance form (Eq. (3.5/6a)) and the limiting case $y_1 = 0$ cannot be represented in the "equivalent voltage source" impedance form (Eq. (3.5/5a)), all three limiting cases $z_1 = 0$, $y_1 = 0$, $r_1 = 0$, can be given in the scattering form, Eq. (3.5/3). Only the special case $z_1 = y_1 = -1$, of no practical interest, cannot be covered with the scattering form. However, this case can be described in the wave representation by the inverse scattering form

$$a_1 = v_1 b_1 + a_{Q1} \qquad (3.5/10a)$$

with

$$v_1 = r_1^{-1} \quad (3.5/10b) \qquad \text{and} \qquad a_{Q1} = -r_1^{-1} b_{Q1} \quad (3.5/10c)$$

For $z_1 = y_1 = -1$, then, $v_1 = 0$ and $a_{Q1} = u_{Q1}/2 = i_{Q1}/2$. The special case of the primary wave source ($r_1 = 0$, $z_1 = y_1 = -1$), the most important for microwave engineering, has the equation of state

$$b_1 = b_{Q1} \qquad (3.5/11a)$$

in the scattering form, and the equations of state

$$u_1 = i_1 + u_{Q1} \qquad (3.5/11b)$$
$$i_1 = u_1 + i_{Q1}. \qquad (3.5/11c)$$

in the impedance and conductance forms, respectively. Here, too, a comparison of the three Eqs. (3.5/11) indicates the advantage of the wave representation over the voltage/current representations.

* * *

3.52 General Linear n-Port Sources

If a single-frequency microwave generator to which a linear 1-port source can be assigned is connected with a suitable waveguide junction which can be considered a linear, time-independent and sourceless (n+1)-port network a system involving a source is generally obtained again which we can designate as a linear n-port source. We assume for the present that such a linear n-port source can be described in the wave representation by the scattering form

$$\begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_k \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1k} & \cdots & s_{1n} \\ s_{21} & s_{22} & & & & \\ \cdot & & \ddots & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ s_{k1} & & & s_{kk} & & \\ \cdot & & & & \ddots & \\ \cdot & & & & & \\ \cdot & & & & & \\ s_{n1} & & & & & s_{nn} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_k \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} + \begin{pmatrix} b_{Q1} \\ b_{Q2} \\ \cdot \\ \cdot \\ \cdot \\ b_{Qk} \\ \cdot \\ \cdot \\ \cdot \\ b_{Qn} \end{pmatrix} \qquad (3.5/12a)$$

or, abbreviated,

$$B = S\,A + B_Q \qquad (3.5/12b)$$

Here, $\underline{A}$ and $\underline{B}$ are the vectors, already explained in Section 3.42, for the arriving and departing waves, respectively, in the reference planes of the n-port network; S is the scattering matrix for the n-port source. The quantity

$$B_Q = \begin{pmatrix} b_{Q1} \\ b_{Q2} \\ \cdot \\ \cdot \\ \cdot \\ b_{Qk} \\ \cdot \\ \cdot \\ \cdot \\ b_{Qn} \end{pmatrix} \qquad (3.5/13)$$

is the vector -- independent of $\underline{A}$ and $\underline{B}$ -- for the departing source wave values b_{Qk}. The n components b_{Qk} here are all components of the same frequency, and are locked in phase with one another. Their individual amplitudes (peak values) $|b_{Qk}|$ and phases arc (b_{Qk}) are established by the interconnection of the 1-port source and the (n+1)-port network. In special cases, one or more source waves can vanish. However, we always speak of an n-port source, if, for n ports, not all components of column vector $\underline{B}_Q$ are identically equal to zero in the domain of definition of the n-port network.

We can now disregard the special interconnection considered above and generally designate generator junctions with the previously discussed properties as linear n-port sources if they can be described by the equations of state (3.5/12) or an appropriate equivalent form (cf. Table 7). In the following, we always assume here that all source parameters derive from the same physical single-frequency generation mechanism and are therefore of the same frequency; it is not necessary, however, that the "primary generator" be physically separable as its own junction (1-port source). Examples of such n-port gate sources with inseparable 1-port sources in microwave engineering include l-waveguide generators which are accessible via non-dominant-mode lines (e.g. circular waveguides) or, generally, those which generate oscillations with a frequency above the second-lowest cutoff frequency of the waveguide, and, generally, all N-waveguide generators (N > 1).

Table 7

General linear n-port gate source in four representations and conversions of the characteristic parameters of the n-port source

		a Wellen-Darstellungen		b Spannungs/Strom-Darstellungen	
		c B-Quelle	A-Quelle	u-Quelle	i-Quelle
a Wellen-Darstellungen	S B_Q	$B = SA + B_Q$	$S = V^{-1}$ $B_Q = -V^{-1}A_Q$	$S = (z+E)^{-1}(z-E)$ $B_Q = (z+E)^{-1}u_Q$	$S = -(y+E)^{-1}(y-E)$ $B_Q = -(y+E)^{-1}i_Q$
	V A_Q	$V = S^{-1}$ $A_Q = -S^{-1}B_Q$	$A = VB + A_Q$	$V = (z-E)^{-1}(z+E)$ $A_Q = (z-E)^{-1}u_Q$	$V = -(y-E)^{-1}(y+E)$ $A_Q = -(y-E)^{-1}i_Q$
b Spannungs/Strom-Darstellungen	z u_Q	$z = (E-S)^{-1}(E+S)$ $u_Q = 2(E-S)^{-1}B_Q$	$z = -(E-V)^{-1}(E+V)$ $u_Q = 2(E-V)^{-1}B_Q$	$u = zi + u_Q$	$z = y^{-1}$ $u_Q = -y^{-1}i_Q$
	y i_Q	$y = (E+S)^{-1}(E-S)$ $i_Q = -2(E+S)^{-1}B_Q$	$y = -(E+V)^{-1}(E-V)$ $i_Q = 2(E+V)^{-1}A_Q$	$y = z^{-1}$ $i_Q = -z^{-1}u_Q$	$i = yu + i_Q$

Key: a. Wave representations
b. Voltage/current representations
c. Source

If the n-port source is closed off with suitable absorbers, then, in accordance with Definition 11, all n waves a_k arriving at the source are zero, and thus each departing wave b_k is equivalent to the corresponding source wave: $b_k = b_{Qk}$. This is again a measurement rule for determining source waves. We obtain the total real power delivered to all absorbers from the n-port source here by calculating the transported power for each departing wave and then summing these[11]:

$$P_p^- = \sum_{k=1}^{n} P_{pk}^- = \sum_{k=1}^{n} \tfrac{1}{2}\,|b_k|^2 = \sum_{k=1}^{n} \tfrac{1}{2}\,|b_{Qk}|^2 . \qquad (3.5/14a)$$

We can also write the last summation as the scalar product (row times column) of the source wave column vector, thereby obtaining:

$$P_p^- = \tfrac{1}{2}\,(b_{Q1}\; b_{Q2} \ldots b_{Qn})^* \begin{pmatrix} b_{Q1} \\ b_{Q2} \\ \cdot \\ \cdot \\ \cdot \\ b_{Qn} \end{pmatrix} = \tfrac{1}{2}\, B_Q^{*T} B_Q . \qquad (3.5/14b)$$

This is at the same time the total power which the the n-port source delivers to a sourceless n-port network if the scattering matrix of the source is equivalent to the null matrix (cf. Section 4.64).

Making use of the designations used with the sourceless n-port network, we likewise wish to call the coefficients $s_{k\ell}$ of the scattering matrix for the n-port source reflection coefficients (for $k = \ell$) or transmission coefficients (for $k \neq \ell$). However, they cannot be determined in the same straightforward

[11] Even if several wave types exist in a waveguide, the total transported power is obtained as the summation of individual powers, due to the orthogonality of the individual structure functions (cf. Section 2.33 and Section 3.2).

manner as in the case of the sourceless n-port network. We shall not cover the possible methods for determining the s_{kl} experimentally until Section 4.75.

If the n-port source is associated with a generator junction of N waveguides, the state-variable vectors $\underline{A}$, $\underline{B}$ and $\underline{B}_Q$ and, correspondingly, the scattering matrix can be broken down into N groups assigned to the individual waveguides just as in the case of the sourceless n-port network. The overall scattering matrix then contains N^2 submatrices: N reflection matrices $\underline{S}_{KK}$ and $N^2 - N$ transmission matrices $\underline{S}_{KL}$ $(K \neq L)$. If all transmission matrices $\underline{S}_{KL}$ $(K \neq L)$ are null matrices, then the N waveguide connections are uncoupled. If all transmission coefficients $s_{k\ell}$ $(k \neq \ell)$ are zero, the n-port source breaks down into n mutually independent 1-port sources; finally, if all scattering coefficients are zero, we are dealing with n mutually independent 1-port primary wave sources[12].

Just as the 1-port source can, an n-port source in the wave representation can also be described formally by the inverse scattering form

$$A = V B + A_Q \qquad (3.5/15)$$

with the inverse scattering matrix $\underline{V} = \underline{S}^{-1}$, as well as by the scattering form, if $\underline{S}$ is not singular. $\underline{A}_Q$ is the column vector for the arriving source waves here. In addition, equations of state for the n-port source

$$u = z\, i + u_Q \qquad (3.5/16)$$

and

$$i = y\, u + i_Q \qquad (3.5/17)$$

[12] In the special case in which all $s_{k\ell} = 0$ $(k \neq \ell)$, i.e. n independent 1-port sources, the conditions of equal frequency for the source waves can be eliminated. When the frequencies of the n 1-port sources are different, it is then necessary, for example, to carry out power calculations n times and add the results.

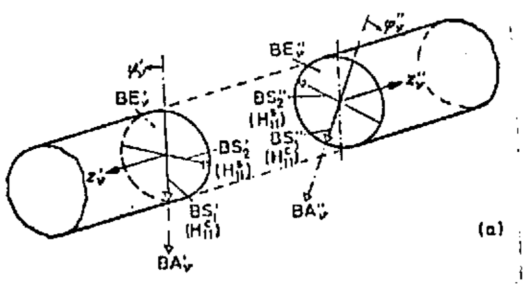
can be set up as an extension of equations (3.5/5a) and (3.5/6a) in the voltage/ current representation. Here, u_Q and i_Q are the column vectors for the source voltages and source currents, respectively; the remaining column vectors and matrices have the same meaning as for sourceless n-port networks in the voltage/ current representation. In Table 7, which itself is laid out on a matrix scheme, we find the four representations of the general linear n-port source in the heavily outlined principal diagonal fields and, in the secondary diagonal fields, the characteristic parameters of a mode of representation expressed by means of the characteristic parameters of another mode of representation.

* * *

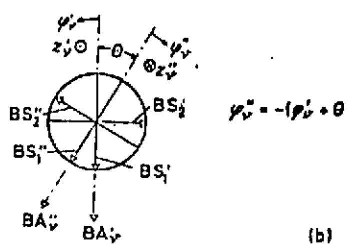
4.133 Switching Group With Rotational Transformation

In addition to the two previously treated cases without type mixing, there also exists the case of a compatible combination with the mixing of directionally degenerate types in special rotationally symmetrical waveguides. We wish to study the coupling relations which apply here, using a simple but practically important example, and therefore consider a compatible combination via a pair of circular waveguides in the two-wave range as shown in Fig. 4/12.

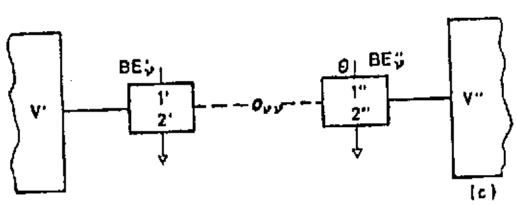
In accordance with Section 3.81 and Fig. 4/12a, let port 1' be assigned to the H_{11}^{c} wave with reference structure $BS_{1'}$, and port 2' to the H_{11}^{s} wave with reference structure $BS_{2'}$, in reference plane $BE_{\nu'}$, and, correspondingly, port 1" to the H_{11}^{c} wave with $BS_{1''}$ and port 2" to the H_{11}^{s} wave with $BS_{2''}$ in reference plane $BE_{\nu''}$. The reference axes $BA_{\nu'}$ and $BA_{\nu''}$ of the two reference planes are not parallel or antiparallel now, however, as in the cases covered above; rather, they are oriented so that they enclose angle θ as shown in Fig. 4/12b. Due to this rotation of the reference axes and thus of the reference structures, a departing H_{11}^{c} wave (port 1') in reference $BE_{\nu'}$ will produce both an arriving H_{11}^{c} wave (port 1") and a departing H_{11}^{s} wave (port 2") in reference plane $BE_{\nu''}$.



(a) Waveguides with rotated reference structure



(b) Coordinate system and reference structure



(c) Equivalent circuit diagram

Fig. 4/12. Example of a connection with type mixing.

In order to derive the coupling relations, we compile the expressions for the transverse electric fields of all arriving and departing H_{11} waves in $BE_{\nu'}$ ($z_{\nu'} = 0$) and $BE_{\nu''}$ ($z_{\nu''} = 0$), respectively. We obtain

$$E_{t1}^{+\prime}(z_\nu' = 0) = t_1' \sqrt{Z_{F1}'}\, a_1', \qquad (4.1/22a)$$
$$E_{t1}^{-\prime}(z_\nu' = 0) = t_1' \sqrt{Z_{F1}'}\, b_1', \qquad (4.1/22b)$$
$$E_{t2}^{+\prime}(z_\nu' = 0) = t_2' \sqrt{Z_{F2}'}\, a_2', \qquad (4.1/22c)$$
$$E_{t2}^{-\prime}(z_\nu' = 0) = t_2' \sqrt{Z_{F2}'}\, b_2' \qquad (4.1/22d)$$

and

$$E_{t1}^{+\prime\prime}(z_v''=0) = t_1''\sqrt{Z_{F1}''}\,a_1'', \qquad (4.1/23a)$$
$$E_{t1}^{-\prime\prime}(z_v''=0) = t_1''\sqrt{Z_{F1}''}\,b_1'', \qquad (4.1/23b)$$
$$E_{t2}^{+\prime\prime}(z_v''=0) = t_2''\sqrt{Z_{F2}''}\,a_2'', \qquad (4.1/23c)$$
$$E_{t2}^{-\prime\prime}(z_v''=0) = t_2''\sqrt{Z_{F2}''}\,b_2''. \qquad (4.1/23d)$$

When they are combined, i.e., if the two reference planes coincide ($z_{\nu'} = -z_{\nu''}$), we must then equate the superposition of the two departing waves in $BE_{\nu''}$ to superposition of the two arriving waves in $BE_{\nu'}$:

$$E_{t1}^{-\prime\prime}(z_v''=0) + E_{t2}^{-\prime\prime}(z_v''=0) = E_{t1}^{+\prime}(z_v'=0) + E_{t2}^{+\prime}(z_v'=0). \quad (4.1/24a)$$

Correspondingly, we obtain the following for the opposite direction:

$$E_{t1}^{-\prime}(z_v'=0) + E_{t2}^{-\prime}(z_v'=0) = E_{t1}^{+\prime\prime}(z_v''=0) + E_{t2}^{+\prime\prime}(z_v''=0). \quad (4.1/24b)$$

Due to the degeneration of the H_{11}^{c} and H_{11}^{s} waves, and since both waveguides form a compatible pair, we obtain

$$Z_{F1}' = Z_{F2}' = Z_{F1}'' = Z_{F2}''. \qquad (4.1/25)$$

If we now substitute Eqs. (4.1/22) and (4.1/23) into the vectorial coupling relations (4.1/24), taking Eq. (4.1/25) into consideration, we obtain

$$t_1''\,b_1'' + t_2''\,b_2'' = t_1'\,a_1' + t_2'\,a_2' \qquad (4.1/26a)$$
$$t_1'\,b_1' + t_2'\,b_2' = t_1''\,a_1'' + t_2''\,a_2''. \qquad (4.1/26b)$$

Both structure functions $t_{1'}, t_{2'}$ and structure functions $t_{1''}, t_{2''}$ each form an orthogonal pair with the same eigenvalue. According to Section 2.33, we can convert one pair into the other pair by means of an orthogonal transformation, for example

$$\begin{pmatrix} t_1' \\ t_2' \end{pmatrix} = \begin{pmatrix} o_{11} & o_{12} \\ o_{21} & o_{22} \end{pmatrix} \begin{pmatrix} t_1'' \\ t_2'' \end{pmatrix}. \qquad (4.1/27a)$$

If we now express functions $t_{1'}$ and $t_{2'}$ in Eq. (4.1/26a) with $t_{1''}$ and $t_{2''}$, with the aid of transformation (4.1/27a), we obtain

$$t_1'' b_1'' + t_2'' b_2'' = (o_{11} t_1'' + o_{12} t_2'') a_1' + (o_{21} t_1'' + o_{22} t_2'') a_2'$$

or, after collecting the coefficients of the same structure functions,

$$t_1'' b_1'' + t_2'' b_2'' = t_1'' (o_{11} a_1' + o_{21} a_2') + t_2'' (o_{12} a_1' + o_{22} a_2')$$

Since $t_{1''}$ is orthogonal to $t_{2''}$, it thus follows that

$$b_1'' = o_{11} a_1' + o_{21} a_2' \qquad (4.1/28a)$$
$$b_2'' = o_{12} a_1' + o_{22} a_2'. \qquad (4.1/28b)$$

We thus have a set of scalar coupling relations for connecting ports 1',2' with ports 1",2".

The transformation matrix in Eq. (4.1/27a) we call $\underline{o}$. Since it is orthogonal, $\underline{o}^{-1} = \underline{o}^T$. We can now solve Eq. (4.1/27a) very easily with

$$\begin{pmatrix} t_1'' \\ t_2'' \end{pmatrix} = \begin{pmatrix} o_{11} & o_{21} \\ o_{12} & o_{22} \end{pmatrix} \begin{pmatrix} t_1' \\ t_2' \end{pmatrix} \qquad (4.1/27b)$$

and thereby express structure functions $t_{1''}, t_{2''}$ with $t_{1'}, t_{2'}$ in Eq. (4.1/26b). In a corresponding manner, we then obtain the other set of scalar coupling relations with

$$b_1' = o_{11} a_1'' + o_{12} a_2'', \qquad (4.1/29a)$$
$$b_2' = o_{21} a_1'' + o_{22} a_2'' \qquad (4.1/29b)$$

We can easily see, by comparison with coupling relations set up earlier, that the switching group matrices are in this case equivalent to the transformation matrix $\underline{o}$ or its transpose $\underline{o}^T$.

The coefficients of the transformation matrix can be determined in the following manner. The known structure functions for the H_{11}^{c} and H_{11}^{s} waves in $BE_{\nu'}$ and $BE_{\nu''}$ (e.g. as in Table 2) are substituted into the transformation equation, e.g. (4.1/27a). The right and left sides of Eq. (4.1/27a) can be compared component-wise after the coordinate transformation

$$\varphi'' = -(\varphi' + \theta)$$
$$e_{\varphi}' = -e_{\varphi}''$$

By comparing coefficients we then obtain

$$o_{11} = \cos\theta, \quad o_{12} = -\sin\theta,$$
$$o_{21} = -\sin\theta, \quad o_{22} = -\cos\theta.$$

Thus transformation matrix $\underline{o}$ is symmetrical, i.e., $\underline{o} = \underline{o}^{T}$. Since it is also orthogonal ($\underline{o}^{-1} = \underline{o}^{T}$), we also have $\underline{o}^{-1} = \underline{o}$.

We again collect the arriving- and departing-wave parameters for ports 1', 2' and 1'',2'' to form column vectors $\underline{A}'$,$\underline{B}'$ and $\underline{A}''$,$\underline{B}''$, respectively, and in place of Eqs. (4.1/29) and (4.1/28) we write

$$B' = G_{o}^{(',")} A'', \qquad (4.1/30a)$$
$$B'' = G_{o}^{(",')} A' \qquad (4.1/30b)$$

with the switching group matrices

$$G_{o}^{(',")} = G_{o}^{(",')} = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}. \qquad (4.1/31)$$

The subscript o is meant to indicate that these switching group matrices characterize a connection with type mixing with respect to an orthogonal transformation. For the special angles of rotation $\theta = 0°$ and $180°$, only ports 1' and 1'', and 2' and 2'', respectively, are connected together; the switching group matrices are then diagonal and characterize a pure interchange of signs (cf. Eqs. (4.1/12) through (4.1/15)). If angle θ is $90°$ or $270°$, port 1' is only connected with port 2'' and port 2' only with port 1''; this is then a special case of an interchange of port numbers. In general, the connection is represented by the equivalent circuit diagram shown in Fig. 4/12c.

If a compatible combination is produced via a circular waveguide pair with twisted reference axes and with more than just the two H_{11} waves, the switching group matrices can be easily generalized with the aid of the example discussed. The directionally degenerate wave types are then coupled via Eq. (4.1/31) to analogous, orthogonal submatrices which are arranged along the principal diagonals of the switching group matrices, corresponding to port numbering. Of course, this method can also be applied in a corresponding manner to coaxial-line connections in the multiple-wave range.

4.134 General Properties of the Switching Group Matrices

In a general compatible combination of two junctions or networks, the three interchange operations which have been considered separately till now can occur in combination. Connection of the two port groups TG' and TG'' of m ports each can then be described by suitable switching group matrices whose properties we now wish to derive from generally valid laws governing connection. We first collect the two switching matrix equations (4.1/16a,b) or (4.1/19a,b) or (4.1/30a,b) to form one matrix equation

$$\begin{pmatrix} \boldsymbol{B}' \\ \boldsymbol{B}'' \end{pmatrix} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{G}^{(','')} \\ \boldsymbol{G}^{('',')} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{A}' \\ \boldsymbol{A}'' \end{pmatrix} \qquad (4.1/32a)$$

and abbreviate it with

$$\boldsymbol{B} = \boldsymbol{T}_S \boldsymbol{A} \qquad (4.1/32b)$$

Column vectors $\underline{A}$ and $\underline{B}$ are column vectors for the arriving and departing waves, respectively, with respect to junctions V' and V". However, we can also conceive of the "switching point," i.e. the totality of reference planes which coincide in the combination, as a proper "switching" junction to which a 2m-port network can be assigned. Relative to this 2m-port network, $\underline{A}$ is then the column vector for departing and $\underline{B}$ the vector for arriving waves. If we invert Eq. (4.1/32b)

$$\underline{A} = \underline{T}_S^{-1}\,\underline{B}$$

and substitute $\underline{A} = \underline{B}_S$, $\underline{B} = \underline{A}_S$ and $\underline{T}_S^{-1} = \underline{S}_S$, we obtain

$$\underline{B}_S = \underline{S}_S \underline{A}_S, \qquad (4.1/33)$$

the equation of state for the 2m-port network in customary scattering form. We can now characterize the properties of the 2m-port network by means of the 2m-row scattering matrix

$$S_S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \qquad (4.1/34)$$

with m-row submatrices $\underline{S}_{ij}$, using the known criteria in Section 3.6.

Due to the compatibility of the combination and the definition of switching junctions, the 2m-port network has the following characteristics:

1. The m ports of port group TG' are not coupled with one another; for this reason,

$$\underline{S}_{11} = \underline{0}. \qquad (4.1/35a)$$

The m ports in port group TG'' are likewise not coupled with one another; for this reason,

$$\underline{S}_{22} = \underline{0}. \qquad (4.1/35b)$$

2. The 2m-port network is reciprocal; i.e., its scattering matrix $\underline{S}$ is symmetrical. Thus

$$\underline{S}_S = \underline{S}_S^T \qquad (4.1/35c)$$

and therefore

$$\underline{S}_{21} = \underline{S}_{12}^T. \qquad (4.1/36d)$$

Characterization with one transmission matrix, e.g. $\underline{S}_{12}$, is thus sufficient.

3. Since the switching junction has no spatial dimensions, transmission from ports in port group TG' to ports in port group TG'' and vice versa occurs without phase displacement. Transmission matrix $\underline{S}_{12}$ is therefore real,

$$\underline{S}_{12} = \underline{S}_{12}^*, \qquad (4.1/35e)$$

and, due to Eqs. (4.1/35a,b), scattering matrix $\underline{S}$ is also real,

$$\underline{S}_S = \underline{S}_S^*. \qquad (4.1/35f)$$

4. The 2m-port network is neutral; i.e. its scattering matrix $\underline{S}_S$ is unitary,

$$\underline{S}_S^{*T}\, \underline{S}_S = \underline{E}. \qquad (4.1/35g)$$

From Eqs. (4.1/35g), (4.1/35f) and (4.1/35c), taken together, it then follows that

$$\underline{S}_S \underline{S}_S = \underline{E} \qquad (4.1/35h)$$

or

$$\underline{S}_S = \underline{S}_S^{-1}.$$

The scattering matrix is thus involutory, i.e., equal to its inverse.

Since we had introduced scattering matrix $\underline{S}_S$ as the inverse of transformation matrix $\underline{T}_S$, we thus also have

$$\underline{T}_S = \underline{S}_S. \qquad (4.1/36)$$

This means that all properties of $\underline{S}_S$ defined by Eq. (4.1/35) also apply to $\underline{T}_S$. In particular,

$$\underline{G}^{(',")} = \underline{S}_{12} \quad \text{and} \quad \underline{G}^{(",')} = \underline{S}_{21}$$

and, from Eq. (4.1/35d),

$$\underline{G}^{(",')} = \underline{G}^{(',")T}. \qquad (4.1/37)$$

Connection of the two port groups TG' and TG" to form one switching group is thus adequately characterized by one switching group matrix $\underline{G}$. We write

$$G^{(',")} = G \qquad (4.1/38a)$$

and

$$G^{(",')} = G^T \qquad (4.1/38b)$$

and then obtain

$$T_S = \begin{pmatrix} O & G \\ G^T & O \end{pmatrix} \qquad (4.1/39)$$

and, in place of Eq. (4.1/32a),

$$\begin{pmatrix} B' \\ B'' \end{pmatrix} = \begin{pmatrix} O & G \\ G^T & O \end{pmatrix} \begin{pmatrix} A' \\ A'' \end{pmatrix}. \qquad (4.1/40)$$

We wish to call matrix $\underline{T}_S$ the transformation matrix for calculating connections in the scattering form. The connection is described here by equation of state (4.1/40). From Eqs. (4.1/36) and (4.1/35), switching group matrix $\underline{G}$ is real and orthogonal, i.e.,

$$G^* = G, \qquad (4.1/41a)$$
$$G^{-1} = G^T. \qquad (4.1/41b)$$

If junctions V' and V" are characterized by wave "chain" matrixes, and if the connection of port groups TG' and TG" is to be calculated in the cascade form, then coupling relations of the form

$$A'(B'')$$
and
$$B'(A'')$$

are necessary for this purpose. From Eq. (4.1/40), we obtain

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} (G^T)^{-1} & O \\ O & G \end{pmatrix} \begin{pmatrix} B'' \\ A'' \end{pmatrix}.$$

by rearrangement and inversion. By interchanging the operations of transposition and inversion and using Eq. (4.1/41b), we obtain $(\underline{G}^T)^{-1} = (\underline{G}^{-1})^T = (\underline{G}^T)^T = \underline{G}$, and from this the equation of state for the connection in the cascade form

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} G & O \\ O & G \end{pmatrix} \begin{pmatrix} B'' \\ A'' \end{pmatrix}. \tag{4.1/42}$$

The matrix

$$T_C = \begin{pmatrix} G & O \\ O & G \end{pmatrix} \tag{4.1/43}$$

we call the transformation matrix for calculating connections in the cascade form.

* * *

4.22 Transformation of a 1-Port Source

We next study the transformation of a 1-port source by a sourceless 2-port network, and for this purpose we open the network in Fig. 4/15 at switching point 2,2'. Let the original 1-port source with port 1' be described by equation of state

$$b_{1'} = r_{1'} a_{1'} + b_{Q'}. \tag{4.2/8}$$

At switching point 1,1', the parameters of state are coupled by the switching equations

$$b_1 = p_{11'} a_1', \tag{4.2/9a}$$
$$b_1' = p_{1'1} a_1 \tag{4.2/9b}$$

In addition, we again have the equations of state (4.2/1) for the 2-port network at our disposal. The 1-port source resulting from the connection, as shown in Fig. 4/17, with available port 2, is assumed to be described by the equation of state

$$b_2 = r_2\ a_2 + b_{Q2}. \qquad (4.2/10)$$

The (internal) reflection factor r_2 and source wave b_{Q2} from the tranformed source or equivalent wave source must now be determined with the aid of Eqs. (4.2/1), (4.2/8) and (4.2/9). For this purpose, we replace the parameters of state $a_{1'}, b_{1'}$ for port 1' in Eq. (4.2/8) with the parameters of state a_1, b_1 for port 1, using switching equations (4.2/9), and solve for b_1:

$$b_1 = r_1'^{-1}\, a_1 - r_1'^{-1}\, p_{11'}\, b_{Q1'}.$$

We thereby eliminate b_1 in 2-port network equation (4.2/1a) and solve for a_1:

$$a_1 = (r_1'^{-1} - s_{11})^{-1}\, s_{12}\, a_2 + (r_1'^{-1} - s_{11})^{-1}\, r_1'^{-1}\, p_{11'}\, b_{Q1'}.$$

If we now substitute this result in 2-port network equation (4.2/1b) and factor out a_2, we obtain

$$b_2 = [s_{22} + s_{21}\,(r_1'^{-1} - s_{11})^{-1}\, s_{12}]\, a_2 + s_{21}\,(r_1'^{-1} - s_{11})^{-1}\, r_1'^{-1}\, p_{11'}\, b'_{Q1}. \qquad (4.2/11\,a)$$

By comparison with Eq. (4.2/10), we then find

$$r_2 = s_{22} + s_{21}\,(r_1'^{-1} - s_{11})^{-1}\, s_{12} \qquad (4.2/11\,b)$$

for the transformed (internal) reflection factor and

$$b_{Q2} = s_{21}\,(r_1'^{-1} - s_{11})^{-1}\, r_1'^{-1}\, p_{11'}\, b_{Q1'}. \qquad (4.2/11\,c)$$

for the transformed source wave.

This result is interesting in two regards:

The reflection factor given in Eq. (4.2/5b) for a transformed, sourceless 1-port network is, as can be seen by comparison with Eq. (4.2/11b), formally identical to the reflection factor for a transformed 1-port source. Thus the transformation of a sourceless 1-port network is included in the more general transformation of a 1-port source as the special case in which $b_Q = 0$. We also see that the reference-arrow reversal at the switching point involved with an interchange of signs as determined by polarity parameter $p_{\nu\nu}$, only has an effect on the transformation of the parameters of state in this case, e.g. on the source wave, for example, but not on the transformation of the operator parameters, such as the reflection factor in this case.

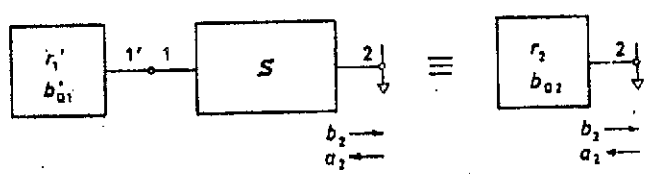


Fig. 4/17. Connection of a 1-port source with a sourceless 2-port network, and equivalent 1-port source.

For practical application, we suitably transform relationships (4.2/11b,c) and write the following: $r_{1'} = r_{i1}$, $r_2 = r_{i2}$ and $b_{Q1'} = b_{Q1}$; then

$$b_2 = r_{i2}\, a_2 + b_{Q2} \tag{4.2/12a}$$

with

$$r_{i2} = s_{22} + \frac{s_{21}\, s_{12}\, r_{i1}}{1 - s_{11}\, r_{i1}} \tag{4.2/12b}$$

and

$$b_{Q2} = \frac{s_{21}}{1 - s_{11}\, r_{i1}}\, p_{11} \cdot b_{Q1}. \tag{4.2/12c}$$

If the original source is a primary wave source, i.e. $r_{i1} = 0$, then

$$r_{12} = s_{22}$$

and

$$b_{Q2} = s_{21}\, p_{11} \cdot b_{Q1}.$$

If the transforming 2-port network is an ideal loss-free line, i.e. $s_{11} = s_{22} = 0$ and $s_{12} = s_{21} = \exp(-j\beta\ell)$, then

$$r_{12} = e^{-j2\beta l}\, r_{11}$$

and

$$b_{Q2} = e^{-j\beta l}\, p_{11} \cdot b_{Q1}.$$

The transformation thus become particularly simple if the product $s_{11}r_{i1}$ disappears in the numerator of Eqs. (4.2/12b,c). This is obviously the case if either the original source ($r_{i1} = 0$) and/or the port connected to the source ($s_{11} = 0$) are matched.

4.23 Chain Configurations of 2-Port Networks

If two or more 2-port networks are put in a chain as shown in Fig. 4/18 or 4/19, a 2-port network again exists. We now consider the problem of determining the characteristic parameters of the resultant 2-port network from the parameters from the individual 2-port units. This task can be treated in the wave representation both in the scattering form and in the chain form.

We first study the connection of two 2-port networks as in Fig. 4/18 in the scattering form, and base our discussion on the general case in which the 2-port network with ports 1 and 2 represents a 2-port source which can be described with the equations of state

$$b_1 = s_{11}\, a_1 + s_{12}\, a_2 + b_{Q1}, \qquad (4.2/13\,a)$$

$$b_2 = s_{21}\, a_1 + s_{22}\, a_2 + b_{Q2} \qquad (4.2/13\,b)$$

The other 2-port network is assumed sourceless and is described by the equations of state

$$b_3 = s_{33}\, a_3 + s_{34}\, a_4, \tag{4.2/14a}$$
$$b_4 = s_{43}\, a_3 + s_{44}\, a_4 \tag{4.2/14b}$$

Ports 2 and 3 are joined when the connection is made as shown in Fig. 4/18; ports 1 and 4 form the port of the resulting 2-port network, which then generally again represents a 2-port source. To describe the chain combination in the scattering form, it is desirable to consider the two unconnected 2-port networks together as a 4-port network to collect ports 1 and 4 in one group and ports 2 and 3 in another group. With this breakdown, we then obtain the following from Eqs. (4.2/13) and (4.2/14) as the equation of state for the 4-port network

$$\begin{pmatrix} b_1 \\ b_4 \\ \hdashline b_2 \\ b_3 \end{pmatrix} = \left(\begin{array}{cc:cc} s_{11} & 0 & s_{12} & 0 \\ 0 & s_{44} & 0 & s_{43} \\ \hdashline s_{21} & 0 & s_{22} & 0 \\ 0 & s_{34} & 0 & s_{33} \end{array}\right) \begin{pmatrix} a_1 \\ a_4 \\ \hdashline a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_{Q1} \\ 0 \\ \hdashline b_{Q2} \\ 0 \end{pmatrix} \tag{4.2/15a}$$

or, with the subvectors

$$\boldsymbol{A}_1 = \begin{pmatrix} a_1 \\ a_4 \end{pmatrix}, \qquad \boldsymbol{B}_1 = \begin{pmatrix} b_1 \\ b_4 \end{pmatrix}, \qquad \boldsymbol{B}_{Q1} = \begin{pmatrix} b_{Q1} \\ 0 \end{pmatrix},$$
$$\boldsymbol{A}_2 = \begin{pmatrix} a_2 \\ a_3 \end{pmatrix}, \qquad \boldsymbol{B}_2 = \begin{pmatrix} b_2 \\ b_3 \end{pmatrix}, \qquad \boldsymbol{B}_{Q2} = \begin{pmatrix} b_{Q2} \\ 0 \end{pmatrix}$$

and the submatrices

$$\boldsymbol{S}_{11} = \begin{pmatrix} s_{11} & 0 \\ 0 & s_{44} \end{pmatrix}, \qquad \boldsymbol{S}_{12} = \begin{pmatrix} s_{12} & 0 \\ 0 & s_{43} \end{pmatrix},$$
$$\boldsymbol{S}_{21} = \begin{pmatrix} s_{21} & 0 \\ 0 & s_{34} \end{pmatrix}, \qquad \boldsymbol{S}_{22} = \begin{pmatrix} s_{22} & 0 \\ 0 & s_{33} \end{pmatrix}$$

in abbreviated notation,

$$B_1 = S_{11} A_1 + S_{12} A_2 + B_{Q1},$$
$$B_2 = S_{21} A_1 + S_{22} A_2 + B_{Q2}. \qquad (4.2/15b)$$

The connecting of ports 2 and 3 can be described, on the basis of Eqs. (4.1/39) and (4.1/40), by the equation

$$B_2 = T_S A_2 \qquad (4.2/16)$$

with the tranformation matrix

$$T_S = \begin{pmatrix} 0 & p_{23} \\ p_{32} & 0 \end{pmatrix} \qquad (4.2/17)$$

The polarity parameters $p_{23} = p_{32}$ are either +1 here if the reference arrows for ports 2 and 3 are in the same direction -- as shown in Fig. 4/18 -- or -1 if the reference arrows are in opposite directions.

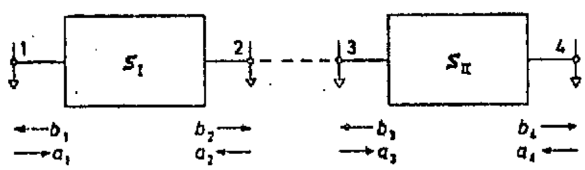


Fig. 4/18. Diagram for calculating the scattering matrix of a chain of two 2-port networks

The parameters of state for ports 2 and 3, which are to be connected, i.e. subvectors $\underline{A}_2$ and $\underline{B}_2$, must now be eliminated again and system of equations (4.2/15) solved for $\underline{B}_1(\underline{A}_1, \underline{B}_{Q1}, \underline{B}_{Q2})$. Similarly to the method applied in Section 4.21, we thus replace $\underline{B}_2$ in the second line of Eq. (4.2/15b) with $\underline{A}_2$, using switching equation (4.2/16), and solve for $\underline{A}_2$

$$A_2 = (T_S - S_{22})^{-1} S_{21} A_1 + (T_S - S_{22})^{-1} B_{Q2}.$$

We then substitute this expression into the first line of Eq. (4.2/15b) and obtain

$$B_1 = [S_{11} + S_{12}(T_S - S_{22})^{-1} S_{21}]\, A_1 + S_{12}(T_S - S_{22})^{-1} B_{Q2} + B_{Q1} \tag{4.2/18a}$$

or

$$B_1 = S'\,A_1 + B_Q'. \tag{4.2/18b}$$

The scattering matrix $\underline{S}'$ for the resultant 2-port source here is

$$S' = S_{11} + S_{12}(T_S - S_{22})^{-1} S_{21} \tag{4.2/19a}$$

and, after inserting the submatrices and multiplying out,

$$S' = \begin{pmatrix} s_{11} + \dfrac{s_{12}\,s_{21}\,s_{33}}{1 - s_{22}\,s_{33}} & \dfrac{p_{23}\,s_{12}\,s_{34}}{1 - s_{22}\,s_{33}} \\ \dfrac{p_{23}\,s_{43}\,s_{21}}{1 - s_{22}\,s_{33}} & s_{44} + \dfrac{s_{43}\,s_{34}\,s_{22}}{1 - s_{22}\,s_{33}} \end{pmatrix}. \tag{4.2/19b}$$

For the source wave column vector $\underline{B}_{Q'}$, we obtain

$$B_Q' = S_{12}(T_S - S_{22})^{-1} B_{Q2} + B_{Q1} \tag{4.2/20a}$$

or

$$B_Q' = \begin{pmatrix} b_{Q1}' \\ b_{Q4}' \end{pmatrix} = \begin{pmatrix} \dfrac{s_{12}\,s_{33}}{1 - s_{22}\,s_{33}}\, b_{Q2} + b_{Q1} \\ \dfrac{p_{23}\,s_{43}}{1 - s_{22}\,s_{33}}\, b_{Q2} \end{pmatrix}. \tag{4.2/20b}$$

If the 2-port network with ports 1, 2 is sourceless, then $b_{Q1} = b_{Q2} = 0$, and thus $\underline{B}_{Q'} = \underline{0}$. The scattering matrix for the resultant sourceless 2-port network is then likewise determined by Eq. (4.2/19).

Another special case is also included in this computation, namely that of a 2-port source with a load on one side at port 2: We then consider $s_{33} = r_{2A}$ to be the initial reflection factor and, with $s_{43} = 0$, $s_{34} = 0$, obtain the characteristic values for the resultant 1-port source from the coefficients in Eq. (4.2/19b) or (4.2/20b) which then remain.

According to Eq. (4.2/19), the scattering matrix for the resultant 2-port network generally becomes quite simple when the reflection coefficients s_{22} and s_{33} for the connected ports vanish. If the original 2-port networks are matched at all ports, i.e., $s_{11} = s_{22} = s_{33} = s_{44} = 0$, then the resultant 2-port network is also matched at both ports, i.e. $s_{1'1'} = s_{4'4'} = 0$. If the original 2-port networks are reciprocal, then the resultant 2-port network is also reciprocal.

The scattering-form computational method used to characterize the chain coupling of two 2-port networks can be extended in a corresponding manner to chain configurations of more than two such networks. In many cases, particularly if all 2-port networks are sourceless, it is more advantageous to carry out the calculations in a chain form, however. We now consider a chain of q sourceless 2-port networks as shown in Fig. 4/19. The individual 2-port networks are assumed to be characterized by their wave chain matrices $\underline{C}$ in accordance with Eq. (3.4/24), as defined in Section 3.44.

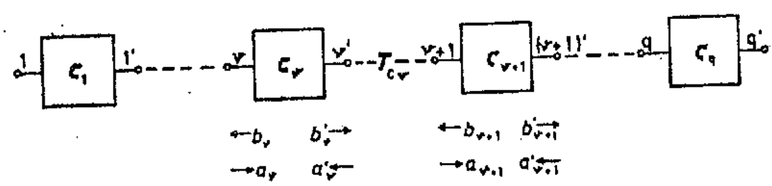


Fig. 4/19. Diagram for calculating the wave chain matrix for a chain of q 2-port networks

We first single out two neighboring 2-port networks ν, $\nu+1$ from the overall chain; these are assumed to be described by the equations of state

$$\begin{pmatrix} b_\nu \\ a_\nu \end{pmatrix} = C_\nu \begin{pmatrix} a_\nu' \\ b_\nu' \end{pmatrix} \qquad (4.2/21)$$

and

$$\begin{pmatrix} b_{\nu+1} \\ a_{\nu+1} \end{pmatrix} = C_{\nu+1} \begin{pmatrix} a_{\nu+1}' \\ b_{\nu+1}' \end{pmatrix} \qquad (4.2/22)$$

The connecting of port ν' and port ν+1 can be represented, according to Eq. (4.1/42), by

$$\begin{pmatrix} a_{\nu}' \\ b_{\nu}' \end{pmatrix} = T_{C\nu} \begin{pmatrix} b_{\nu+1} \\ a_{\nu+1} \end{pmatrix} \qquad (4.2/23)$$

with transformation matrix

$$T_{C\nu} = p_{\nu',\nu+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (4.2/24)$$

Here, $p_{\nu',\nu+1}$ is the polarity parameter for the port connection. By combining Eqs. (4.2/21) and (4.2/22) with the aid of switching equation (4.2/23), we obtain the equation of state for the two-member chain as

$$\begin{pmatrix} b_{\nu} \\ a_{\nu} \end{pmatrix} = C_{\nu}\, T_{C\nu}\, C_{\nu+1} \begin{pmatrix} a_{\nu+1}' \\ b_{\nu+1}' \end{pmatrix}. \qquad (4.2/25)$$

Similarly, we obtain the following for the chain of q 2-port networks

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = C_1\, T_{C1}\, C_2\, T_{C2} \dots C_{\nu}\, T_{C\nu} \dots C_{q-1}\, T_{Cq-1}\, C_q \begin{pmatrix} a_q' \\ b_q' \end{pmatrix}. \qquad (4.2/26)$$

Since the transformation matrix, according to Eq. (4.2/24), is a scalar matrix in this case, it can be separated out of the matrix product. For the wave chain matrix $\underline{C}$ for the q-member 2-port network chain we then obtain

$$C = \prod_{\nu=1}^{q-1} (p_{\nu',\nu+1}) \prod_{\nu=1}^{q} C_{\nu}. \qquad (4.2/27)$$

The product over the polarity parameters is either +1 or -1. If all individual 2-port networks are equivalent to one another, the product over the chain matrices is equal to the qth power of $\underset{\sim}{C}^q_\nu$, which can easily be determined, for example, by diagonalization ([B3], [B4]).

* * *

4.62 Connecting Two General Subnetworks in a Chain

As shown in Fig. 4/54, we now break the microwave network down into subnetwork N_α, with port groups TG_1 and TG_2, and subnetwork N_β, with port groups TG_3 and TG_4. Port group TG_1 contains n ports; port group TG_4, p ports; and port groups TG_2 and TG_3 each contain m ports which are assumed to form a compatible switching group SG.

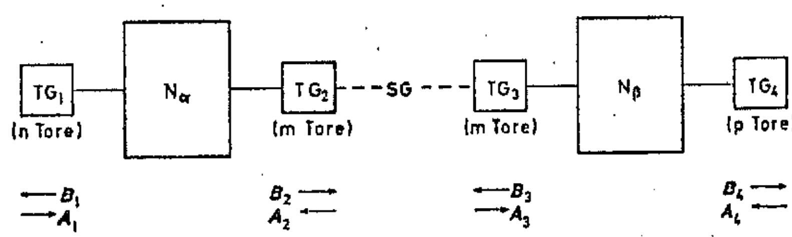


Fig. 4/54. Chain coupling of two subnetworks N_α and N_β

Key: Tore = ports

Subnetwork N_α is again assumed to be represented generally by an (n+m)-port source with the equations of state (4.6/1)

$$\begin{aligned} B_1 &= S_{11} A_1 + S_{12} A_2 + B_{Q1}, \\ B_2 &= S_{21} A_1 + S_{22} A_2 + B_{Q2} \end{aligned}$$

Subnetwork N_β in general represents an (m+p)-port source which is assumed to be described by the equations of state

$$B_3 = S_{33} A_3 + S_{34} A_4 + B_{Q3}, \tag{4.6/6a}$$
$$B_4 = S_{43} A_3 + S_{44} A_4 + B_{Q4} \tag{4.6/6b}$$

Here, $\underline{S}_{33}$ is an m×m submatrix, and $\underline{S}_{44}$ is a p×p submatrix. The generally rectangular submatrices $\underline{S}_{34}$ and $\underline{S}_{43}$ have m rows and p columns, and p rows and m columns, respectively.

Let switching group SG, made up of port groups TG_2 of N_α and TG_3 of N_β, again be characterized by switching equation (4.6/ 3). Port groups TG_2 and TG_3 are saturated by connecting the two subnetworks; the resulting network, as a generalized chain configuration of the two subnetworks, is accessible via port groups TG_1 and TG_4.

In order to make calculations for this general chain, we mix the equations of state (4.6/1) and (4.6/6) for the two separate subnetworks, whereby the column vectors of state for the resultant port groups TG_1 and TG_4 or port groups TG_2 and TG_3, to be connected together, are collected in new column vectors as shown in the following scheme:

$$\begin{pmatrix} B_1 \\ B_4 \\ B_2 \\ B_3 \end{pmatrix} = \left(\begin{array}{cc:cc} S_{11} & O & S_{12} & O \\ O & S_{44} & O & S_{43} \\ \hdashline S_{21} & O & S_{22} & O \\ O & S_{34} & O & S_{33} \end{array}\right) \begin{pmatrix} A_1 \\ A_4 \\ A_2 \\ A_3 \end{pmatrix} + \begin{pmatrix} B_{Q1} \\ B_{Q4} \\ B_{Q2} \\ B_{Q3} \end{pmatrix}. \tag{4.6/7}$$

With the abbreviations

$$A_x = \begin{pmatrix} A_1 \\ A_4 \end{pmatrix} \text{(4.6/8a)}, \quad B_x = \begin{pmatrix} B_1 \\ B_4 \end{pmatrix} \text{(4.6/8b)}, \quad B_{Qx} = \begin{pmatrix} B_{Q1} \\ B_{Q4} \end{pmatrix}, \text{(4.6/8c)}$$
$$A_y = \begin{pmatrix} A_2 \\ A_3 \end{pmatrix} \text{(4.6/8d)}, \quad B_y = \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} \text{(4.6/8e)}, \quad B_{Qy} = \begin{pmatrix} B_{Q2} \\ B_{Q3} \end{pmatrix} \text{(4.6/8f)}$$

and

$$S_{xx} = \begin{pmatrix} S_{11} & O \\ O & S_{44} \end{pmatrix} \quad (4.6/8\,g), \qquad S_{xy} = \begin{pmatrix} S_{12} & O \\ O & S_{43} \end{pmatrix}, \quad (4.6/8\,h)$$

$$S_{yx} = \begin{pmatrix} S_{21} & O \\ O & S_{34} \end{pmatrix} \quad (4.6/8\,i), \qquad S_{yy} = \begin{pmatrix} S_{22} & O \\ O & S_{33} \end{pmatrix} \quad (4.6/8\,j)$$

we then obtain the following in place of Eq. (4.6/7):

$$B_x = S_{xx} A_x + S_{xy} A_y + B_{Qx}, \qquad (4.6/9\,a)$$
$$B_y = S_{yx} A_x + S_{yy} A_y + B_{Qy}. \qquad (4.6/9\,b)$$

With abbreviations (4.6/8d) and (4.6/8e), we can express switching equations (4.6/3) with the transformation equation

$$B_y = T_S A_y \qquad (4.6/10)$$

in which transformation matrix $\underline{T}_S$ from Eq. (4.1/39)

$$T_S = \begin{pmatrix} O & G \\ G^T & O \end{pmatrix}$$

is made up of switching group matrix $\underline{G}$ and an m×m null matrix.

As determined by the numbers of rows and columns of the submatrices, $\underline{S}_{xx}$ is an (n+p)×(n+p) matrix, and $\underline{S}_{yy}$ and $\underline{T}_S$ are 2mx2m matrices. The generally rectangular matrices $\underline{S}_{xy}$ and $\underline{S}_{yx}$ have (n+p) rows and 2m columns, and 2m rows and (n+p) columns, respectively.

Again the problem arises of eliminating the column vectors of state for the connecting port groups TG_2 and TG_3, i.e. the column vectors $\underline{A}_y$ and $\underline{B}_y$. We first substitute Eq. (4.6/10) into Eq. (4.6/9b) and solve for $\underline{A}_y$:

$$A_y = (T_S - S_{yy})^{-1} S_{yx} A_x + (T_S - S_{yy})^{-1} B_{Qy}.$$

We next substitute $\underline{A}_y$ into Eq. (4.6/9a) and, after factoring out $\underline{A}_x$, we obtain

$$B_x = [S_{xx} + S_{xy}(T_S - S_{yy})^{-1} S_{yx}] A_x + S_{xy}(T_S - S_{yy})^{-1} B_{Qy} + B_{Qx} \tag{4.6/11 a}$$

or, abbreviated

$$B_x = \underline{\overline{S}} A_x + \overline{B}_Q. \tag{4.6/11 b}$$

Here, scattering matrix $\underline{\overline{S}}$ for the chain configuration of the two subnetworks is given by

$$\overline{S} = S_{xx} + S_{xy}(T_S - S_{yy})^{-1} S_{yx} \tag{4.6/12 a}$$

and the resultant source wave column vector $\underline{\overline{B}}_Q$ by

$$\overline{B}_Q = B_{Qx} + S_{xy}(T_S - S_{yy})^{-1} B_{Qy} \tag{4.6/12 b}$$

We write the inverse $\underline{I}$ of the 2m×2m "supermatrix" $\underline{T}_S - \underline{S}_{yy}$ in the form

$$(T_S - S_{yy})^{-1} = \begin{pmatrix} -S_{22} & G \\ G^T & -S_{33} \end{pmatrix}^{-1} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}. \tag{4.6/13}$$

If we invert the supermatrix, we obtain the following for m×m submatrices $\underline{I}_{\nu\mu}$:

$$I_{11} = G\, S_{33}(G - S_{22}\, G\, S_{33})^{-1}, \tag{4.6/14 a}$$
$$I_{12} = (G^T - S_{33}\, G^T S_{22})^{-1}, \tag{4.6/14 b}$$
$$I_{21} = (G - S_{22}\, G\, S_{33})^{-1}, \tag{4.6/14 c}$$
$$I_{22} = G^T S_{22}(G^T - S_{33}\, G^T S_{22})^{-1}. \tag{4.6/14 d}$$

Due to the orthogonality of the switching group matrix, these inversions become particularly simple if one of the two inherent reflection matrices $\underline{S}_{22}$ or $\underline{S}_{33}$ is a null matrix.

We now designate the resultant scattering matrix $\overline{\underline{S}}$ and source wave vector $\overline{\underline{B}}_Q$, on the basis of gate groups TG_1 and TG_4, through which the resultant network is accessible, as

$$\overline{S} = \begin{pmatrix} \overline{S}_{11} & \overline{S}_{14} \\ \overline{S}_{41} & \overline{S}_{44} \end{pmatrix} \quad (4.6/15a) \qquad \text{and} \qquad \overline{B}_Q = \begin{pmatrix} \overline{B}_{Q1} \\ \overline{B}_{Q4} \end{pmatrix}. \quad (4.6/15b)$$

With the aid of Eqs. (4.6/8), (4.6/13), (4.6/14), we then obtain the following froms Eqs. (4.6/12) for the submatrices defined by Eq. (4.6/15a) for the resultant network:

$$\begin{aligned}
\overline{S}_{11} &= S_{11} + S_{12}\, G\, S_{33}\, (G - S_{22}\, G\, S_{33})^{-1}\, S_{21}\,, && (4.6/16a)\\
\overline{S}_{14} &= S_{12}\, (G^{T} - S_{33}\, G^{T}\, S_{22})^{-1}\, S_{34}\,, && (4.6/16b)\\
\overline{S}_{41} &= S_{43}\, (G - S_{22}\, G\, S_{33})^{-1}\, S_{21}\,, && (4.6/16c)\\
\overline{S}_{44} &= S_{44} + S_{43}\, G^{T}\, S_{22}\, (G^{T} - S_{33}\, G^{T}\, S_{22})^{-1}\, S_{34} && (4.6/16d)
\end{aligned}$$

and the following for the source wave subvectors defined by Eq. (4.6/15b):

$$\begin{aligned}
\overline{B}_{Q1} &= B_{Q1} + S_{12}\, G\, S_{33}\, (G - S_{22}\, G\, S_{33})^{-1}\, B_{Q2} + \\
&\quad + S_{12}\, (G^{T} - S_{33}\, G^{T}\, S_{22})^{-1}\, B_{Q3}\,, && (4.6/16e)\\
\overline{B}_{Q4} &= B_{Q4} + S_{43}\, G^{T}\, S_{22}\, (G^{T} - S_{33}\, G^{T}\, S_{22})^{-1}\, B_{Q3} + \\
&\quad + S_{43}\, (G - S_{22}\, G\, S_{33})^{-1}\, B_{Q2}. && (4.6/16f)
\end{aligned}$$

The transformation problem treated in Section 4.61 is included directly -- with the selected symbols and port-number assignments to the different port groups -- as a special case in this second approach. If we close port group TG_4 at all ports without reflection, subnetwork N_β degenerates to a single-port-group network with port group TG_3, which is characterized by $\underline{S}_{33}$ and $\underline{B}_{Q3}$ alone. If we now equate $\underline{S}_3$ with $\underline{S}_{33}$ and $\overline{\underline{S}}_1$ with $\overline{\underline{S}}_{11}$, Eq. (4.6/16a) is converted into (4.6/5a,c) and (4.6/16e) into (4.6/5b,d).

On the other hand, the chain configurations covered here can also be conceived of as a special case of a general transformation as described in Section 4.61: port group TG_4 is transformed via a partial subnetwork of N_α which acts as a through connection, into a partial port group of TG_1. The six equations (4.6/16) are then obtained from Eq. (4.6/5) if the matrices used herein are conceived of as supermatrices which are built up in a suitable manner from the submatrices for the two networks as shown in Fig. 4/54 and the "through connection."

4.63 Chains of Sourceless 2n-Port Ports Symmetrical with Respect to Port Numbers

If a sourceless network can be broken down into two subnetworks N_α and N_β with two port groups each in such a manner that all port groups have the same number of ports and a port group in N_α and a port group in N_β form a compatible switching group, then the chain configuration of both subnetworks can be computed in the wave chain form. It is assumed here that a corresponding wave chain matrix which is suitable for switching calculations can be set up for every subnetwork (cf. Section 3.44).

We now consider the network broken down as in Fig. 4/54 and assume that every port group consists of n ports (m = p = n). Let each subnetwork be represented by a sourceless 2n-port system. In the wave chain form, subnetwork N_α is described by the equation of state

$$\begin{pmatrix} B_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} \qquad (4.6/17)$$

and subnetwork N_β by equation of state

$$\begin{pmatrix} B_3 \\ A_3 \end{pmatrix} = \begin{pmatrix} C_{33} & C_{34} \\ C_{43} & C_{44} \end{pmatrix} \begin{pmatrix} A_4 \\ B_4 \end{pmatrix} \qquad (4.6/18)$$

Port groups TG_2 and TG_3 are assumed to form a compatible switching group which is characterized by switching group matrix $\underline{G}$. The connecting of the two port groups can then be described on the basis of Eq. (4.1/42) in the cascade form by the transformation

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} G & O \\ O & G \end{pmatrix} \begin{pmatrix} B_3 \\ A_3 \end{pmatrix} \qquad (4.6/19)$$

Under the above assumptions, all column subvectors in Eqs. (4.6/17) through (4.6/19) have n rows, and all submatrices are n×n.

The equation of state for the two subnetworks connected in a chain can now be determined very easily by the step-wise elimination of the vectors of state for port groups TG_2 and TG_3 with the aid of transformation (4.6/19): In Eq. (4.6/17), we express the last column supervector with Eq. (4.6/19), and in the latter we express the last supervector with Eq. (4.6/18), obtaining

$$\begin{pmatrix} B_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} G & O \\ O & G \end{pmatrix} \begin{pmatrix} C_{33} & C_{34} \\ C_{43} & C_{44} \end{pmatrix} \begin{pmatrix} A_4 \\ B_4 \end{pmatrix} \qquad (4.6/20a)$$

or, after multiplying out,

$$\begin{pmatrix} B_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} C_{11}\,G\,C_{33} + C_{12}\,G\,C_{43} & C_{11}\,G\,C_{34} + C_{12}\,G\,C_{44} \\ C_{21}\,G\,C_{33} + C_{22}\,G\,C_{43} & C_{21}\,G\,C_{34} + C_{22}\,G\,C_{44} \end{pmatrix} \begin{pmatrix} A_4 \\ B_4 \end{pmatrix}. \qquad (4.6/20b)$$

This method of computation can be extended in a corresponding manner to chains of several subnetworks which can be represented with sourceless 2n-port networks of equal port numbers, symmetrical with respect to port number. If we designate their 2n-rowed wave chain matrices as $\underline{C}_\nu$ and the transformation matrices for the switching groups from Eq. (4.1/43) as $\underline{T}_{C\nu}$, we obtain the following as an extension of Eq. (4.6/20a) for the equation of state of a q-member chain:

$$\begin{pmatrix} B_1 \\ A_1 \end{pmatrix} = C_1 \, T_{C1} \, C_2 \, T_{C2} \ldots C_\nu \, T_{C\nu} \ldots C_{q-1} \, T_{Cq-1} \, C_q \begin{pmatrix} A_{2q} \\ B_{2q} \end{pmatrix} \quad (4.6/21\,a)$$

or

$$\begin{pmatrix} B_1 \\ A_1 \end{pmatrix} = \left(\prod_{\nu=1}^{q-1} C_\nu \, T_{C\nu} \right) C_q \begin{pmatrix} A_{2q} \\ B_{2q} \end{pmatrix}. \qquad (4.6/21\,b)$$

This computational method is of course particularly advantageous when chains of subnetworks of the same type are involved (cf. Section 4.52).

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